

Week 9 Worksheet - Remainder/Little-oh

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

Remainder Term
 $R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$ $0 < \xi \leq x$ (if x positive)
 Definition of $f(x) = o(x^n)$: $\xi \leq 0$ (if x negative)

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = 0 \quad (f(x) \text{ approaches to } 0 \text{ faster than } x^n \text{ when } x \rightarrow 0.)$$

1. Find the 4th degree Taylor polynomial of $T_4 \cos(2x)$, and estimate the error $|\cos(2x) - T_4 \cos(2x)|$ for $|x| \leq 1$.

Use formula, $\cos u = 1 - \frac{(2u)^2}{2!} + \frac{(2u)^4}{4!}$

$$T_4 \cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$$

$$\begin{aligned} f(x) &= -\sin(2x) \cdot 2 & f^{(4)}(x) &= \cos(2x) \cdot 2^4 \\ f'(x) &= -\cos(2x) \cdot 2^2 & f^{(5)}(x) &= -\sin(2x) \cdot 2^5 \\ f''(x) &= \sin(2x) \cdot 2^3 \end{aligned}$$

Error: $n=4$.

ξ is between 0 and x . ~~$0 < \xi \leq x$~~

$$\text{error} = |R_4 f(x)| = \left| \frac{f^{(5)}(\xi)}{5!} x^5 \right| = \frac{x^5}{5!} |\sin(2\xi) \cdot x^5| \leq \frac{x^5}{5!}$$

\downarrow
since $|\sin(2\xi)| \leq 1$
 $|x^5| \leq 1$

2. $f(x) = \sqrt{9+x}$

- (a) Compute $T_1 f(x)$, and estimate $\sqrt{10}$ using your result.

- (b) Show that the error is less than $1/216$.

- (c*) Repeat with $n=2$. That is, compute $T_2 f(x)$, and estimate $|\sqrt{10} - T_2 f(1)|$.

(a) One can calculate $T_1 f(x)$ directly by definition,
 or use the formula for $(1+x)^a$.

Here, I'll calculate by definition.

$$\begin{aligned} n=1, \quad f(x) &= (x+9)^{\frac{1}{2}} \Rightarrow f(0) = 3 \\ a=0, \quad f'(x) &= \frac{1}{2}(x+9)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2} \cdot 3^{-\frac{1}{2}} = \frac{1}{6} \end{aligned}$$

$$\Rightarrow T_1 f(x) = 3 + \frac{1}{6}x$$

$$\sqrt{10} = f(1) \approx T_1 f(1) = 3 + \frac{1}{6} = \frac{19}{6}$$

$$(b) \quad f'(x) = -\frac{1}{4}(x+9)^{-\frac{3}{2}}$$

$$\begin{aligned} \text{error} &= |R_1 f(x)| = \left| \frac{f''(\xi)}{2} x^2 \right| \Big|_{x=1} \quad \xi \text{ is between 0 and } 1 \\ &= \left| \frac{f''(1)}{2} \cdot 1 \right| = \left| -\frac{1}{4} \cdot \frac{1}{2} \cdot (1+9)^{-\frac{3}{2}} \right| \leq \frac{1}{8} \cdot \frac{1}{9^{\frac{3}{2}}} \\ &= \frac{1}{516} \end{aligned}$$

$$(c) \quad n=2, \quad a=0 \quad f''(0) = -\frac{1}{4} \cdot 9^{-\frac{3}{2}} = -\frac{1}{4} \cdot \frac{1}{27} = -\frac{1}{108}$$

$$\Rightarrow T_2 f(x) = 3 + \frac{1}{6}x - \frac{1}{108}x^2$$

$$f'''(x) = \frac{3}{8}(x+9)^{-\frac{5}{2}}$$

$$\text{Error} = |f(1) - T_2 f(1)| = |R_2 f(1)| = \left| \frac{f'''(\xi)}{3!} 1^3 \right|$$

$$= \left| \frac{1}{6} \cdot \frac{3}{8} (1+9)^{-\frac{5}{2}} \right| \quad \xi \text{ is between 0 and } 1.$$

$$= \left| \frac{1}{16} \frac{1}{\sqrt{100+81}} \right|$$

$$\leq \frac{1}{16} \cdot \frac{1}{9^{\frac{5}{2}}} = \frac{1}{16} \cdot \frac{1}{3^5}$$

3. Estimate $|R_1 f(x)|$ for $f(x) = x \sin^2 x$ for $|x| \leq 1$.

$$f'(x) = \sin^2 x + x \cdot 2 \sin x \cos x$$

$$f''(x) = 2 \sin x \cos x + 2 \sin x \cos x + 2x \cos^2 x - 2x \sin^2 x = 2 \sin(2x) + 2x \cos(2x)$$

$$\begin{aligned} |R_1 f(x)| &= \left| \frac{f''(\xi)}{2!} x^2 \right| = \left| \frac{2 \sin(2\xi) + 2\xi \cos(2\xi)}{2} x^2 \right| \\ &\leq \left| \frac{2+2 \cdot 1}{2} \cdot 1^2 \right| = 2 \end{aligned}$$

ξ between 0, x.

4. True or False. (most from past exams)

F (a) $\sin(x) - x - \frac{x^3}{3!} = o(x^4)$

T (b) $\sin(x^2) - x^2$ is $o(x^4)$.

T (c) $R_n f(x) = o(x^n)$

F (d) $T_n f(x) = o(x^n)$

T (e) $(x^2 + x^3)^2 = o(x^3)$

F (f) $(x \cos x - x)$ is $o(x^5)$.

T (g) $e^x - \sqrt{1+2x} = o(x)$ $e^x - [1 + \binom{\frac{1}{2}}{1}(2x)] + o(x) = o(x)$

T (h) $o(3x) = o(x)$

F (i) $o(x^2) + o(x) = o(x^2)$

F (j) $o(x) - o(x) = o(x^2)$

T (k) $R_4 \sin x = \sin x - \left(x - \frac{x^3}{3!}\right)$

F (l) $R_4 e^x = e^x - (1 + x + x^2 + x^3 + x^4)$

F (m) Let $f(x)$ and $g(x)$ be functions whose Taylor series exist. Then for any n we have

$$T_n[f(x)g(x)] = (\underbrace{T_n f(x)}_{\text{degree } n})(\underbrace{T_n g(x)}_{\text{degree } n})$$

$$\text{degree } n \cdot n = n^2$$

5. Find the possible values of k such that $\sqrt[3]{1+x^2} = 1 + o(x^k)$.

$$f(x) = (1+x^2)^{\frac{1}{3}} \quad \text{use formula } (1+u)^a = 1 + \binom{a}{1} u + \binom{a}{2} u^2 + \dots$$

$$= 1 + \binom{\frac{1}{3}}{1} x^2 + \binom{\frac{1}{3}}{2} x^4 + \dots$$

From question, find k s.t. $f(x) - 1 = \binom{\frac{1}{3}}{1} x^2 + \dots = o(x^k) \Rightarrow k = 0, 1$